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STABILITY ANALYSIS OF A NONLINEAR DIFFERENTIAL EQUATION SYSTEM USING THE FIRST APPROXIMATION METHOD

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Abstract

This article examines Lyapunov's first approximation method, which is one of the important approaches for analyzing the stability of equilibrium states in nonlinear systems of differential equations. A linearized model of the system is constructed in the neighborhood of the equilibrium point, and the stability conditions are determined using the eigenvalues of the Jacobian matrix. The theoretical results are illustrated through an example, demonstrating the practical significance of the method.

Keywords

Differential equations, stability theory, Lyapunov method, first approximation, equilibrium point, Jacobian matrix, eigenvalues, asymptotic stability, nonlinear system, linearization.

Annotatsiya. Ushbu maqolada nochiziqli differensial tenglamalar sistemalarining muvozanat holati turg‘unligini tekshirishning muhim usullaridan biri bo‘lgan Lyapunovning birinchi yaqinlashish usuli o‘rganiladi. Sistemaning muvozanat nuqtasi atrofida chiziqlashtirilgan modeli qurilib, hosil bo‘lgan Yakobi matritsasining xos qiymatlari yordamida turg‘unlik shartlari aniqlanadi. Nazariy natijalar na’muna misol orqali yoritilib, usulning amaliy ahamiyati ko‘rsatib beriladi.

Kalit so‘zlar. Differensial tenglamalar, turg‘unlik nazariyasi, Lyapunov usuli, birinchi yaqinlashish, muvozanat nuqtasi, Yakobi matritsasi, xos qiymatlar, asimptotik turg‘unlik, nochiziqli sistema, chiziqlashtirish.

Kirish. Differensial tenglamalar nazariyasida sistemalarning turg‘unligini o‘rganish muhim masalalardan biri hisoblanadi. Fizika, mexanika, biologiya,

iqtisodiyot va texnikada uchraydigan ko‘plab jarayonlar differensial tenglamalar sistemalari orqali modellashtiriladi. Bunday sistemalar uchun muvozanat holatlarining mavjudligi va ularning turg‘unligini aniqlash katta amaliy ahamiyatga ega.

Turg‘unlik nazariyasining asoschisi hisoblangan Aleksandr Mixaylovich Lyapunov tomonidan ishlab chiqilgan usullar bugungi kunda ham keng qo‘llanilmoqda. Xususan, Lyapunovning birinchi yaqinlashish usuli nochiziqli sistemalarni muvozanat nuqtasi atrofida chiziqlashtirishga asoslanadi. Ushbu usul yordamida murakkab nochiziqli sistemaning turg‘unligi unga mos chiziqli sistemaning xossalari orqali aniqlanadi.

Mazkur maqolada birinchi yaqinlashish usulining nazariy asoslari bayon qilinadi hamda muvozanat nuqtasining turg‘unligi Yakobi matritsasining xos qiymatlari yordamida tekshiriladi. Shuningdek, usulning qo‘llanilishiga oid misol ko‘rib chiqiladi.

Asosiy qism. Aytaylik, bizga quyidagi

$$\frac{dx(t)}{dt} = f(x(t)) \qquad (1)$$

ko‘rinishidagi muxtor differensial tenglamalar sistemasi berilgan bo‘lsin. Bu yerda $t \in R$, $x(t) = (x_1(t), \ldots, x_n(t))^T$, $f(x) = (f_1(x), \ldots, f_n(x))^T$ vektor-funksiyalar bo‘lib, $f(x)$ biror $G \subset R_n^x$ sohada uzluksiz differensiallanuvchi vektor-funksiya. Bundan tashqari $x(t) = 0$ nuqta (1) sistemaning muvozanat nuqtasi, ya’ni $f(0) = 0$ bo‘lsin. U holda $f(x)$ vektor-funksiyani $x = 0$ nuqta atrofida Teylor formulasidan foydalanib quyidagicha

$$f(x) = Ax + \bar{o}(|x|) \qquad (2)$$

yozish mumkin. Bunda qoldiq had $\bar{o}(|x|)$ Peano ko‘rinishida olingan:

$$A = \left\| \frac{\partial f_i(0)}{\partial x_j} \right\|, \quad i, j = \overline{1,n}; \quad \bar{o}(|x|) \to 0, \quad |x| = \sqrt{\sum_{j=1}^{n} |x_j|^2} \to 0.$$

1-teorema. Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta_j$, $j = \overline{1,m}$, $m \le n$ xos qiymatlari

$$\alpha_j = \operatorname{Re} \lambda_j < 0, \quad j = \overline{1,n} \qquad (3)$$

tengsizlikni qanoatlantirsa, u holda (1) nochiziqli muxtor sistemaning $x(t) \equiv 0$ yechimi asimptotik turg‘un bo‘ladi.

Isbot. Berilgan (1) muxtor sistemani $x=0$ nuqtaning atrofida quyidagicha yozish mumkin:

$$\frac{dx(t)}{dt} = Ax(t) + r(x). \qquad (4)$$

Bu yerda

$$r(x) = \overline{o}(|x|). \qquad (5)$$

Berilgan (1) muxtor differensial tenglamalar sistemasining ushbu

$$x(0) = x^0, \quad x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T \qquad (6)$$

Boshlang‘ich shartni qanoatlantiruvchi $x(t)$ yechimi $\forall t \in [0;\infty)$ oraliqda aniqlangan. Bu $x(t)$ yechimni

$$x(t) = e^{tA}x^0 + \int_0^t e^{(t-\tau)A} r(x(\tau))d\tau \qquad (7)$$

ko‘rinishida yozish mumkin. Ma’lumki, bir jinsli chiziqli differensial tenglamalar sistemasi uchun quyidagi

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x^0$$

Koshi masalasining $x(t, x^0)$ yechimini ushbu

$$x(t, x^0) = e^{tA}x^0 \qquad (8)$$

ko‘rinishida yozish mumkin edi.

1-lemma. Agar A matritsaning barcha $\lambda_j = \alpha_j + i\beta_j$, $j = \overline{1,m}$, $m \le n$ xos qiymatlari

$$\alpha_j = \operatorname{Re}\lambda_j < 0, \quad j = \overline{1,m}, \quad m \le n$$

tengsizlikni qanoatlantirsa, u holda shunday $\exists \mu > 0$, $M > 0$ sonlari topilib,

$$\| e^{tA} \| \le Me^{-\mu t} \qquad (9)$$

baho o‘rinli bo‘ladi.

Isbot. Aytaylik, $\mu > 0$ sonini shunday tanlaymizki, natijada ushbu $\operatorname{Re}\lambda_j < -\mu < 0$ tengsizlik bajarilsin. U holda

$$\operatorname{Re}\lambda_j + \mu \le -\nu, \quad \nu > 0, \quad j = \overline{1,m}, \quad m \le n \qquad (10)$$

o‘rinli bo‘ladi. Ma’lumki, e^{tA} matritsaning har bir $a_{ij}(t)$ elementi uchun quyidagi

$$a_{ij}(t) = \sum_{k=1}^{m} P_k^{(i,j)}(t)e^{\lambda_k t}, \quad i, j = \overline{1,n}$$

ko‘rinish o‘rinli bo‘ladi. Bu yerda $P_k^{(i,j)}(t)$ ko‘phad.

Endi (10) tengsizlikdan foydalanib

$$|P_k^{(i,j)}(t)e^{(\lambda_k+\mu)t}|=|P_k^{(i,j)}(t)e^{(\mathrm{Re}\,\lambda_k+\mu)t}|\le|P_k^{(i,j)}(t)|e^{-\nu t}\to 0,\quad t\to\infty$$

munosabatni olamiz. Bundan

$$|P_k^{(i,j)}(t)e^{(\lambda_k+\mu)t}|\le c_{ij},\quad \forall t>0$$

kelib chiqadi. Bu esa o‘z navbatida

$$|P_k^{(i,j)}(t)e^{\lambda_k t}|\le c_{i,j}e^{-\mu t},\quad \forall t>0 \qquad (11)$$

ekanligini bildiradi. Endi (8) formuladan foydalanib quyidagi

$$|x(t,x^0)|\le\|e^{tA}\|\cdot|x^0|=|x^0|\cdot\sqrt{\sum_{i,j=1}^{n}|a_{ij}(t)^2|}\le|x^0|e^{-\mu t}m\left(\sum_{i,j=1}^{n}|c_{ij}|^2\right)^{\frac{1}{2}}=Me^{-\mu t}\cdot|x^0|,$$

ya’ni

$$\|e^{tA}\|\le Me^{-\mu t},\quad \mu>0$$

bahoni olamiz. +

1-teoremani isbotlashda davom etamiz.

Yuqoridagi (5) bahoda

$$r(x)=\overline{o}(|x|),\quad |x|\to 0$$

bo‘lgani uchun, ixtiyoriy $\forall\varepsilon>0$ soni uchun shunday $\exists\delta>0$ soni topilib, $|x|<\delta$ bajarilganda

$$|r(x)|\le\varepsilon|x|$$

bo‘lishini inobatga olsak, (7) munosabatdan

$$|x(t)|\le|e^{tA}x^0|+\int_0^t|e^{(t-\tau)A}\cdot r(x(\tau))|d\tau\le\|e^{tA}\|\cdot|x^0|+\int_0^t\|e^{(t-\tau)A}\|\cdot|r(x(\tau))|d\tau$$

$$\le Me^{-\mu t}|x^0|+\varepsilon M\int_0^t e^{-\mu(t-\tau)}|x(\tau)|d\tau$$

baho kelib chiqadi. Bu yerda ushbu

$$u(t)=e^{\mu t}|x(t)|$$

belgilashdan foydalansak,

$$u(t)\le M|x^0|+\varepsilon M\int_0^t u(\tau)d\tau$$

tengsizlik hosil bo‘ladi. $u(t)$ funksiyaga Gronulla lemmasini qo‘llasak,

$$u(t) \le M \mid x^0 \mid e^{\varepsilon M t}, \quad \forall t > 0$$

baho kelib chiqadi. Bundan esa

$$\mid x(t) \mid \le M \mid x^0 \mid e^{(\varepsilon M - \mu)t}, \quad \forall t > 0$$

kelib chiqadi. Bu bahodan foydalanib, yetarli kichik $\varepsilon > 0$ larda (masalan $0 < \varepsilon < \frac{\mu}{M}$ bo′lganda)

$$\mid x(t) \mid \to 0, \quad t \to +\infty$$

munosabatni topamiz. Bu esa (1) sistema $x(t) \equiv 0$ yechimining asimptotik turg‘un ekanligini ko‘rsatadi. +

2-teorema. Agar A matritsa $\operatorname{Re}\lambda > 0$ tengsizlikni qanoatlantiruvchi kamida bitta xos qiymatga ega bo‘lsa, u holda (1) sistemaning $x(t) \equiv 0$ yechimi turg‘un bo‘lmaydi (noturg‘un bo‘ladi).

Endi yuqorida keltirilgan nazariy ma’lumotlarni amalda qo‘llab, misol yechish bn shug‘ullanamiz.

Misol. Quyidagi sistemaning trivial yechimini turg‘unlikka tekshiring:

$$\begin{cases} \dot{x} = \ln\left(4y + e^{-3x}\right), \\ \dot{y} = 2y - 1 + \sqrt[3]{1 - 6x}. \end{cases}$$

Yechish: Avvalo trivial yechim mavjudligini tekshiramiz:

$$\ln(4 \cdot 0 + e^0) = \ln 1 = 0,$$

$$2 \cdot 0 - 1 + \sqrt[3]{1 - 0} = 0.$$

Demak, $(x, y) = (0, 0)$ sistemaning muvozanat nuqtasi hisoblanadi.

Teylor formulasi yordamida chiziqli qismini ajratamiz.

Birinchi tenglama uchun

$$e^{-3x} = 1 - 3x + O(x^2),$$

shuning uchun

$$4y + e^{-3x} = 1 - 3x + 4y + O(x^2 + y^2).$$

Ma’lumki,

$$\ln(1 + u) = u + O(u^2),$$

demak

$$\ln(4y+e^{-3x}) = -3x+4y+\psi_1(x,y),$$

bu yerda $\psi_1(x,y) = O(x^2+y^2).$

Ikkinchi tenglama uchun

$$(1+u)^{1/3} = 1+\frac{u}{3}+O(u^2).$$

Bu yerda $u = -6x$, shuning uchun

$$\sqrt[3]{1-6x} = 1-2x+O(x^2).$$

Natijada

$$2y-1+\sqrt[3]{1-6x} = -2x+2y+\psi_2(x,y),$$

bu yerda $\psi_2(x,y) = O(x^2+y^2).$

Shunday qilib, sistema

$$\begin{cases} \dot{x} = -3x+4y+\psi_1(x,y), \\ \dot{y} = -2x+2y+\psi_2(x,y). \end{cases}$$

ko‘rinishga keladi.

Chiziqli qismning matritsasi $A = \begin{pmatrix} -3 & 4 \\ -2 & 2 \end{pmatrix}.$

Xos qiymatlarni topamiz:

$$\begin{vmatrix} -3-\lambda & 4 \\ -2 & 2-\lambda \end{vmatrix} = 0.$$

Bundan

$$(-3-\lambda)(2-\lambda)+8 = 0,$$

$$\lambda^2+\lambda+2 = 0.$$

Diskriminant $D = 1-8 = -7.$

Shuning uchun

$$\lambda_{1,2} = \frac{-1\pm i\sqrt{7}}{2}.$$

Demak, $\operatorname{Re}\lambda_{1,2} = -\frac{1}{2} < 0.$

Lyapunovning birinchi yaqinlashish teoremasiga ko‘ra barcha xos qiymatlarning haqiqiy qismlari manfiy bo‘lgani sababli trivial yechim **asimptotik turg‘un** bo‘ladi.

Xulosa. Maqolada nochiziqli differensial tenglamalar sistemalarining turg‘unligini tekshirishda Lyapunovning birinchi yaqinlashish usuli o‘rganildi. Sistema muvozanat nuqtasi atrofida chiziqlashtirilib, hosil bo‘lgan matritsaning xos qiymatlari orqali turg‘unlik haqida xulosa chiqarish mumkinligi ko‘rsatildi. Agar barcha xos qiymatlarning haqiqiy qismlari manfiy bo‘lsa, muvozanat holati asimptotik turg‘un bo‘ladi; kamida bitta xos qiymatning haqiqiy qismi musbat bo‘lsa, sistema noturg‘un bo‘ladi. Ushbu usul sodda va samarali bo‘lib, ko‘plab amaliy masalalarda turg‘unlikni tekshirish imkonini beradi.

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